

GROWTH OF THE NUMBER OF PERIODIC POINTS FOR MEROMORPHIC MAPS

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ABSTRACT. We show that any dominant meromorphic self-map $f : X \rightarrow X$ of a compact Kähler manifold X is an Artin-Mazur map. More precisely, if $P_n(f)$ is the number of its isolated periodic points of period n (counted with multiplicity), then $P_n(f)$ grows at most exponentially fast with respect to n and the exponential rate is at most equal to the algebraic entropy of f . Further estimates are given when X is a surface. Among the techniques introduced in this paper, the h-dimension of the density between two arbitrary positive closed currents on a compact Kähler surface is obtained.

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1. INTRODUCTION

A self-map on a compact differentiable manifold is called an *Artin-Mazur map* if its number of isolated periodic points of period n grows at most exponentially fast with n . Artin and Mazur proved in [1] that such maps are dense in the set of \mathcal{C}^k maps. On the other hand, Kaloshin constructed large families of diffeomorphisms such that the number of isolated periodic points of period n grows faster than any given sequence of integers [23]. In particular, there are maps which do not satisfy Bowen's formula relating the topological entropy and the exponential growth rate of number of periodic points. Furthermore, the dynamical ζ -function, associated with such a map, is not analytic in any neighborhood of zero.

In this paper, we consider the question in the complex setting. Let X be a compact Kähler manifold of dimension k . Let $f : X \rightarrow X$ be a meromorphic self-map of X . We always assume that f is *dominant*, i.e., the image of f contains an open subset of X . [Otherwise, the study of this map is reduced to the case of dominant maps in a lower dimensional manifold. Note also that meromorphic maps may have indeterminacy sets and hence are not continuous in general.] We will show that such a map is always Artin-Mazur and an upper bound for the growth rate of the number of periodic points, in the spirit of Bowen's formula, holds in general. More precise results will be given in the case of dimension $k = 2$.

In order to state our main results, let us recall some basic notions. The map f is holomorphic outside a (possibly empty) analytic set of co-dimension at least 2 which is called *the indeterminacy set* of f and is denoted by $I(f)$. The graph of f over $X \setminus I(f)$ can be compactified to be an irreducible analytic set of dimension k in $X \times X$ and we

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denote this analytic set by $\Gamma(f)$. If π_1, π_2 are the natural projections from $X \times X$ to its factors and $[\Gamma(f)]$ the current of integration on $\Gamma(f)$, we define the pull-back operator f^* on smooth differential (p, q) -forms ϕ by

$$f^*(\phi) := (\pi_1)_*(\pi_2^*(\phi) \wedge [\Gamma(f)]).$$

This is a (p, q) -current defined by an L^1 form which is smooth outside $I(f)$. The operator commutes with $\partial, \bar{\partial}$ and therefore induces a natural pull-back operator $f^* : H^{p,q}(X, \mathbb{C}) \rightarrow H^{p,q}(X, \mathbb{C})$ on the Hodge cohomology group $H^{p,q}(X, \mathbb{C})$ of X .

The iterate of order n of f is defined by $f^n := f \circ \cdots \circ f$ (n times) on a dense Zariski open set and extends to a dominant meromorphic map on X . Since the above current $f^*(\phi)$ is not smooth in general, we cannot iterate the operators on differential forms. One can consider the iterate of f^* acting on Hodge cohomology but we don't have the identity $(f^n)^* = (f^*)^n$ for every f . This well-known phenomenon, observed by Fornæss and Sibony, is a source of difficulties in the dynamical study of f . However, for $0 \leq p \leq k$ and for any fixed norm on cohomology, the norm of $(f^n)^*$ on $H^{p,p}(X, \mathbb{C})$ has a nice behavior : it was shown by Sibony and the first author that the limit

$$d_p(f) := \lim_{n \rightarrow \infty} \|(f^n)^*\|_{H^{p,p}(X, \mathbb{C})}^{1/n}$$

always exists and is a fundamental bi-meromorphic (finite) invariant of f , independent of the choice of the norm on cohomology. This is *the dynamical degree of order p* of f . We always have $d_0(f) = 1$, $d_p(f) \geq 1$, $d_p(f^n) = d_p(f)^n$, and that $p \mapsto \log d_p(f)$ is concave, i.e., $d_p(f)^2 \geq d_{p-1}(f)d_{p+1}(f)$ for $1 \leq p \leq k-1$. The last dynamical degree $d_k(f)$ is also called *the topological degree*. It is equal to the number of points in a generic fiber of f . *The algebraic entropy* of f is defined by

$$h_a(f) := \max_{0 \leq p \leq k} \log d_p(f).$$

The topological entropy of f is always bounded above by the algebraic entropy, see [10, 12, 13, 20, 32] for details.

Let Δ denote the diagonal of $X \times X$ and $\Gamma(f^n)$ the closure of the graph of f^n in $X \times X$. A point $x \in X$ is a *periodic point of period n* of f if $(x, x) \in \Delta \cap \Gamma(f^n)$, and such a point is *isolated* if (x, x) is an isolated point in $\Delta \cap \Gamma(f^n)$. Note that the set of periodic points of period n of f may have positive dimension even when f is a holomorphic map. Denote by $\text{Per}_n(f)$ (resp. $\text{IPer}_n(f)$) the set of all periodic (resp. isolated periodic) points of period n of f . For $x \in \text{IPer}_n(f)$, the local index of f^n at x , denoted by $\nu_x(f^n)$, is the local multiplicity of the intersection $\Delta \cap \Gamma(f^n)$ at (x, x) , see e.g. [26]. Let

$$P_n(f) := \sum_{x \in \text{IPer}_n(f)} \nu_x(f^n)$$

be the number of isolated periodic points of period n counted with multiplicity. The following dynamical ζ -function of f is similar to the one introduced by Artin and Mazur in [1]

$$\zeta_f(z) := \exp \left(\sum_{n \geq 1} P_n(f) \frac{z^n}{n} \right).$$

Here is our first main theorem.

Theorem 1.1. *Let f be a dominant meromorphic self-map on a compact Kähler manifold X . Let $h_a(f)$ be its algebraic entropy and $P_n(f)$ its number of isolated periodic points of period n counted with multiplicity. Then we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(f) \leq h_a(f).$$

In particular, f is an Artin-Mazur map, i.e., its number of isolated periodic points of period n grows at most exponentially fast with n , and therefore, the dynamical ζ -function of f is analytic in a neighborhood of zero.

Note that the above result can be extended to meromorphic correspondences with essentially the same proof, see also [14]. When f has only isolated periodic points, i.e., $\text{IPer}_n(f) = \text{Per}_n(f)$, the number $P_n(f)$ is equal to the intersection number between $\Gamma(f^n)$ and Δ which can be computed, via the Lefschetz fixed point formula, in terms of the trace of the operator $(f^n)^*$ on Hodge cohomology. In this case, the above theorem can be easily obtained. Also, when X is homogeneous, one can move slightly Δ so that its intersection with $\Gamma(f^n)$ is finite and see that the number of isolated intersection points cannot decrease. An upper bound for $P_n(f)$ can be easily obtained in this case.

Note that if $\pi : X' \rightarrow X$ is a bi-meromorphic map between compact Kähler manifolds, then we can lift f to a dominant meromorphic self-map of X' by setting $f' := \pi^{-1} \circ f \circ \pi$. The maps f and f' induce very related dynamical systems but we don't have $P_n(f) = P_n(f')$ in general. In other words, $P_n(f)$ is not a bi-meromorphic invariant of f . So even when X is bi-meromorphic to a homogeneous manifold, one cannot directly reduce the problem to this case.

In order to obtain the theorem in the general case, we will use a recent theory of densities of currents introduced by Sibony and the first author [16]. Roughly speaking, this theory allows us to dilate the coordinates in the normal directions to Δ in $X \times X$ by a factor λ . Taking $\lambda \rightarrow \infty$, the limit of the image of $\Gamma(f^n)$ by the dilation allows us to bound the number of isolated periodic points in terms of the volume of $\Gamma(f^n)$. The last quantity can be estimated using dynamical degrees of f .

We also expect, at least for large families of maps, that the numbers $P_n(f)$ satisfy

$$P_n(f) = e^{nh_a(f)} + o(e^{nh_a(f)}) \quad \text{or equivalently} \quad \lim_{n \rightarrow \infty} e^{-nh_a(f)} P_n(f) = 1.$$

The problem is widely open. Different approaches were introduced to construct a good number of repelling or saddle periodic points for particular families of maps. This allows for obtaining a good lower bound for $P_n(f)$. We refer to [3, 4, 7, 11, 15, 17, 18] for results in this direction.

We will now concentrate in the case where X is a compact Kähler surface. Despite deep knowledges about complex surfaces, the above basic question is still open. We refer to [19, 21, 27, 31] for some results related to this question. When the topological degree of f is dominant, i.e., $d_2(f) > d_1(f)$, the affirmative answer was recently obtained by the authors in [11]. From now on, we assume that f has *minor topological degree*¹ in the sense that $d_2(f) < d_1(f)$. In this case, we have $h_a(f) = \log d_1(f)$. We also assume that f is *algebraically stable* in the sense of Fornæss-Sibony, i.e., $(f^n)^* = (f^*)^n$ as linear maps

¹Such a map is called a map with small topological degree in literature. We think that our terminology is more appropriate.

on $H^{1,1}(X, \mathbb{C})$ for all $n \in \mathbb{N}$. This condition can be checked geometrically. Here is our second main result.

Theorem 1.2. *Let f be a meromorphic self-map with minor topological degree on a compact Kähler surface X . Let $d_1(f)$ denote the first dynamical degree and $P_n(f)$ the number of isolated periodic points of period n of f counted with multiplicity. Assume that f is algebraically stable in the sense of Fornæss-Sibony. Then we have*

$$P_n(f) \leq d_1(f)^n + o(d_1(f)^n) \quad \text{as } n \rightarrow \infty.$$

Note that in this case, the Lefschetz number, i.e., the intersection number between $\Gamma(f^n)$ and Δ , is not difficult to estimate. It is also equal to $d_1(f)^n + o(d_1(f)^n)$, see Lemma 3.5 below. Observe that saddle periodic points are isolated and of multiplicity 1. The following result is an immediate consequence of the last theorem.

Corollary 1.3. *Under the conditions of Theorem 1.2, assume moreover that there is a family A_n of saddle periodic points of period n of f of cardinality $d_1(f)^n + o(d_1(f)^n)$ which are equidistributed with respect to a probability measure μ , i.e.,*

$$\lim_{n \rightarrow \infty} d_1(f)^{-n} \sum_{a \in A_n} \delta_a = \mu,$$

where δ_a stands for the Dirac mass at a . Then the set of all saddle periodic points and the set of all isolated periodic points (counting multiplicity or not) of period n are also of cardinality $d_1(f)^n + o(d_1(f)^n)$ and equidistributed with respect to μ as $n \rightarrow \infty$.

Under technical conditions, by using Pesin's theory, saddle periodic points can be constructed and satisfy the hypothesis of the last corollary. We refer to [7, 18, 22] for more precision on these conditions, see also [24, pp.694-5]. Corollary 1.3 emphasizes the role of the upper bound of the number of isolated periodic points (Theorem 1.2) in their equidistribution property. This upper bound is sometimes overlooked in literature.

Our last main result deals with algebraically stable bi-meromorphic surface maps $f : X \rightarrow X$. To this end, we will make use of the Saito's local index function $\nu_\bullet(f^n) : \text{Per}_n(f) \rightarrow \mathbb{N}$, introduced by Saito [27] and Iwasaki-Uehara [21], which extends the usual local index function on isolated periodic points to all periodic points, see Section 4 below for more details. Define

$$P'_n(f) := \sum_{x \in \text{Per}_n(f)} \nu_x(f^n).$$

Clearly, $P_n(f) \leq P'_n(f)$.

Theorem 1.4. *Let f be a bi-meromorphic self-map on a compact Kähler surface X which is algebraically stable. Assume that its first dynamical degree satisfies $d_1(f) > 1$. Then*

$$P'_n(f) = d_1(f)^n + o(d_1(f)^n) \quad \text{as } n \rightarrow \infty.$$

Note that in dynamics one often assumes that $d_1(f) > 1$ because otherwise the topological entropy of f is zero and, in some sense, the dynamics is poor.

The paper is organized as follows. In Section 2 we recall some background on positive closed currents and the theory of densities for currents. Basic properties of the action of maps on currents and cohomology will be presented in Section 3 together with the proofs of Theorems 1.1 and 1.2. A main ingredient is the theory of densities for currents.

Knowledge on dynamical Green currents of f is used to get sharp upper bounds on $P_n(f)$ in the second theorem. Finally, the case of bi-meromorphic maps together with Saito's and Iwasaki-Uehara's index will be presented in Section 4. This key point in the proof of Theorem 1.4 is the counterpart of Lefschetz fixed point formula in the presence of non-isolated periodic points. Using the theory of tangent currents as well as some recent results on periodic curves, we are able to control the contribution of non-isolated periodic points.

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2. POSITIVE CLOSED CURRENTS AND THEORY OF DENSITIES

In this section, we first recall some basic properties of positive closed currents on a compact Kähler manifold. We then present some properties of tangent currents, a fundamental notion in theory of densities, that will be used in this work. The reader will find more details in [5, 15, 16, 30].

Let X be a compact Kähler manifold of dimension k and ω a fixed Kähler form on X . Denote by $H^{p,q}(X, \mathbb{C})$ the Hodge cohomology group of bi-degree (p, q) of X and define $H^{p,p}(X, \mathbb{R}) := H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{R})$. The cup-product in $\oplus H^*(X, \mathbb{C})$ is denoted by \smile . If T is a closed (p, q) -current, its class in $H^{p,q}(X, \mathbb{C})$ is denoted by $\{T\}$. If T is a positive closed (p, p) -current, its class belongs to $H^{p,p}(X, \mathbb{R})$. Recall that the pseudo-effective cone $H_{psef}^{p,p}(X) \subset H^{p,p}(X, \mathbb{R})$ is the set of cohomology classes of positive closed (p, p) currents. It is closed, convex and salient in the sense that it contains no non-trivial vector subspace. The nef cone $H_{nef}^{1,1}(X)$ is the closure of the cone of all Kähler classes, i.e., classes of Kähler forms. It is contained in the pseudo-effective cone $H_{psef}^{1,1}(X)$. So it is also convex, closed and salient.

If T is a current on X and φ is a test form of complementary degrees, the pairing $\langle T, \varphi \rangle$ denotes the value of T at φ . If T is a positive (p, p) -current on X , we use the following notion of mass for T given by the formula

$$\|T\| := \langle T, \omega^{k-p} \rangle.$$

This mass is equivalent to the usual mass for order 0 currents. When T is positive and closed, its mass depends only on its cohomology class $\{T\}$ in $H^{p,p}(X, \mathbb{R})$. This is a key point in the calculus with positive closed currents. We will write $T \leq T'$ and $T' \geq T$ for two real (p, p) -currents T, T' if $T' - T$ is a positive current. We also write $c \leq c'$ and $c' \geq c$ for $c, c' \in H^{p,p}(X, \mathbb{R})$ when $c' - c$ is the class of a positive closed (p, p) -current, i.e., an element of $H_{psef}^{p,p}(X, \mathbb{R})$. If V is an analytic subset of pure dimension $k - p$ in X , denote by $[V]$ the positive closed (p, p) -current of integration on V and $\{V\}$ its cohomology class in $H^{p,p}(X, \mathbb{R})$. Given a positive closed current T and a point $x \in X$, the Lelong number of T at x is denoted by $\nu(T, x)$.

We recall now basic facts about tangent currents and prove some abstract results which will allow us to bypass Lefschetz fixed point formula in order to bound the number of

periodic points. We will restrict ourselves to the simplest situation that is needed for the present work, see [16] for more details.

Let V be an irreducible submanifold of X of dimension l and $\pi : E \rightarrow V$ be the normal vector bundle of V in X . For a point $a \in V$, if $\text{Tan}_a X$ and $\text{Tan}_a V$ denote respectively the tangent spaces of X and V at a , the fiber $E_a := \pi^{-1}(a)$ of E over a is canonically identified with the quotient space $\text{Tan}_a X / \text{Tan}_a V$. The zero section of E is naturally identified with V . Denote by \overline{E} the natural compactification of E , i.e., the projectivization $\mathbb{P}(E \oplus \mathbb{C})$ of the vector bundle $E \oplus \mathbb{C}$, where \mathbb{C} is the trivial line bundle over V . We still denote by π the natural projection from \overline{E} to V . Denote by A_λ the multiplication by λ on the fibers of E for $\lambda \in \mathbb{C}^*$, i.e., $A_\lambda(u) := \lambda u$ for $u \in E_a$ and $a \in V$. This map extends to a holomorphic automorphism of \overline{E} .

Let V_0 be an open subset of V which is naturally identified with an open subset of the section 0 in E . A diffeomorphism τ from a neighborhood of V_0 in X to a neighborhood of V_0 in E is called *admissible* if it satisfies essentially the following three conditions: the restriction of τ to V_0 is the identity, the differential of τ at each point $a \in V_0$ is \mathbb{C} -linear and the composition of the following maps

$$E_a \hookrightarrow \text{Tan}_a(E) \rightarrow \text{Tan}_a(X) \rightarrow E_a$$

is the identity map on E_a . Here, the morphism $\text{Tan}_a(E) \rightarrow \text{Tan}_a(X)$ is given by the differential of τ^{-1} at a and the other maps are the canonical ones, see [16].

When V_0 is small enough, there are local holomorphic coordinates on a small neighborhood U of V_0 in X so that over V_0 we identify naturally E with $V_0 \times \mathbb{C}^{k-l}$ and U with an open neighborhood of $V_0 \times \{0\}$ in $V_0 \times \mathbb{C}^{k-l}$ (we reduce U if necessary). In this picture, the identity is a holomorphic admissible map. This picture is called a *standard local setting*. Note that a general admissible map is not necessarily holomorphic and there always exist admissible maps for $V_0 := V$, which are rarely holomorphic.

Consider an admissible map τ as above. Let T be a positive closed (p, p) -current on X without mass on V for simplicity. Define

$$T_\lambda := (A_\lambda)_* \tau_*(T).$$

The family (T_λ) is relatively compact on $\pi^{-1}(V_0)$ when $\lambda \rightarrow \infty$: we can extract convergent subsequences for $\lambda \rightarrow \infty$. The limit currents R are positive closed (p, p) -currents on \overline{E} without mass on V . They are V -conic, i.e., $(A_\lambda)_* R = R$ for any $\lambda \in \mathbb{C}^*$, or equivalently, R is invariant by A_λ .

Such a current R depends on the choice of a subsequence of (T_λ) but it is independent of the choice of τ . This property gives us a large flexibility to work with admissible maps. In particular, using global admissible maps, we obtain positive closed (p, p) -currents R on \overline{E} though τ is not holomorphic. It is also known that the cohomology class of R depends on T but does not depend on the choice of R . This class is denoted by $\kappa^V(T)$ and called *the total tangent class* of T with respect to V . The currents R are *the tangent currents* of T along V . The mass of R and the norm of $\kappa^V(T)$ are bounded by a constant times the mass of T .

Let $-h$ denote the tautological $(1, 1)$ -class on \overline{E} . Recall that $\oplus H^*(\overline{E}, \mathbb{C})$ is a free $\oplus H^*(V, \mathbb{C})$ -module generated by $1, h, \dots, h^{k-l}$ (the fibers of \overline{E} are of dimension $k-l$). So we can write **in a unique way**

$$(2.1) \quad \kappa^V(T) = \sum_{j=\max(0, l-p)}^{\min(l, k-p)} \pi^*(\kappa_j^V(T)) \smile h^{j-l+p},$$

where $\kappa_j^V(T)$ is a class in $H^{l-j, l-j}(V, \mathbb{R})$ with the convention that $\kappa_j^V(T) = 0$ outside the range $\max(0, l-p) \leq j \leq \min(l, k-p)$.

The maximal integer s such that $\kappa_s^V(T) \neq 0$ is called *the tangential h-dimension* of T along V (when the tangent currents of T along V vanish, such an integer doesn't exist and we set the tangential h-dimension equal to 0). The class $\kappa_s^V(T)$ is pseudo-effective, i.e., contains a positive closed $(l-s, l-s)$ -current on V . Since we assume that T has no mass on V , we always have $s < k-p$. If ω_V is any Kähler form on V , the tangential h-dimension of T is also equal to the maximal integer $s \geq 0$ such that $R \wedge \pi^*(\omega_V^s) \neq 0$, except when $R = 0$. In particular, when V is reduced to a single point x , we have $s = 0$ and $\kappa_0^x(T) = \nu(T, x)\{x\}$. So the notion of total tangent classes generalizes the notion of Lelong number. The next result may be regarded as the counterpart of Siu's semi-continuity theorem for Lelong numbers.

Proposition 2.1. ([16, Theorem 4.11]) *With the above notation, let T_n and T be positive closed (p, p) -currents on X such that $T_n \rightarrow T$. Let s be an integer at least equal to the tangential h-dimension of T along V . Then $\kappa_j^V(T_n) \rightarrow 0$ for $j > s$ and any limit class of $\kappa_s^V(T_n)$ is pseudo-effective and is smaller than or equal to $\kappa_s^V(T)$.*

The following result will allow us to bound the number of isolated periodic points of a meromorphic map. We always identify the cohomology group $H^{2k}(X, \mathbb{C})$ with \mathbb{C} using the integrals of top degree differential forms on X .

Proposition 2.2. *Let Γ_n be complex subvarieties of pure dimension $k-l$ in X . Assume that there is a sequence of positive numbers d_n such that $d_n \rightarrow \infty$ and $d_n^{-1}[\Gamma_n]$ converges to a positive closed (l, l) -current T on X . Assume also that the tangential h-dimension of T with respect to V is 0 and that $\{T\} \smile \{V\} = c \in \mathbb{R}^+$. Then the number δ_n of isolated points in the intersection $\Gamma_n \cap V$, counted with multiplicity, satisfies $\delta_n \leq cd_n + o(d_n)$ as $n \rightarrow \infty$.*

Proof. The case $c = 1$ has been proved in [11, Proposition 2.1] using the above semi-continuity for densities. The proof there works for all $c \geq 0$. We can also deduce the general case directly from the case $c = 1$. \square

Let T and S be positive closed currents on X . The density between T and S is represented by the tangent currents of $T \otimes S$ along the diagonal Δ of $X \times X$. We will not give here the formal definition of this density and refer to [16] for details. What we will need is the tangential h-dimension of $T \otimes S$ with respect to Δ . We call it *the h-dimension of the density between T and S* . In the special case when S is a positive measure, the density between T and S can be represented by a number which is the integral of the Lelong number $\nu(T, x)$ with respect to the measure S . In this case, the density is always of h-dimension 0 and doesn't vanish if and only if the above integral of $\nu(T, x)$ with respect to S is non-zero. We have the following proposition.

Proposition 2.3. *Let T and S be two positive closed currents on X of bi-degrees (p, p) and (q, q) respectively with $1 \leq p, q \leq k-1$. If S has no mass on the set $\{x \in X, \nu(T, x) > 0\}$, then the h-dimension of the density between T and S is at most equal to $k-q-1$.*

Proof. Consider the positive measure $S' := S \wedge \omega^{k-q}$. By hypothesis, S' has no mass on $\{x \in X, \nu(T, x) > 0\}$. We deduce that the density between T and S' vanishes. Assume by contradiction that the h-dimension of the density between T and S is larger than or equal to $k - q$. We will obtain a contradiction by showing that the density between T and S' doesn't vanish.

Let π_1 and π_2 denote the natural projections from $X \times X$ to its factors. Similarly to the beginning of this section, denote by $\pi : E \rightarrow \Delta$ the normal vector bundle to Δ in $X \times X$ which extends to the natural compactification \overline{E} of E . Let $A_\lambda : \overline{E} \rightarrow \overline{E}$ be the multiplication by λ on the fibers of π . Fix also a sequence $\lambda_n \rightarrow \infty$ so that $(A_{\lambda_n})_* \tau_*(T \otimes S)$ converges to a tangent current R of $T \otimes S$ along Δ which does not depend on the choice of any admissible map τ from a neighborhood of Δ in $X \times X$ to a neighborhood of Δ in E .

If α is any continuous positive closed form on $X \times X$, it is easy to deduce that $(A_{\lambda_n})_* \tau_*((T \otimes S) \wedge \alpha)$ converges to $R \wedge \pi^*(\alpha|_\Delta)$. In particular, we have

$$(A_{\lambda_n})_* \tau_*(T \otimes S') \rightarrow R \wedge \pi^*(\pi_2^*(\omega^{k-q})|_\Delta).$$

If we identify Δ with X in the canonical way, then the last current is equal to $R \wedge \pi^*(\omega^{k-q})$ because $\pi_2^*(\omega^{k-q})|_\Delta$ is identified with ω^{k-q} . Finally, since we assumed that the h-dimension of R is $\geq k - q$, the last current doesn't vanish. It follows that the density between $T \otimes S'$ and Δ doesn't vanish. This is a contradiction we are looking for. \square

The following corollary will be used in the study of meromorphic maps on surfaces. It gives a complete characterization of when the density between two positive closed $(1, 1)$ -currents is of h-dimension 0.

Corollary 2.4. *Let X be a compact Kähler surface. Let T and S be two positive closed $(1, 1)$ -currents on X . Assume that there is no compact analytic curve Y in X such that both T and S have positive mass on Y . Then the density between T and S is of h-dimension 0.*

Proof. Note that by a theorem of Siu, T has positive mass on Y if and only if it has positive Lelong number at each point of Y or equivalently it is equal to the sum of a positive closed current and a positive constant times $[Y]$, see e.g. [5]. Moreover, the set $\{x \in X, \nu(T, x) > 0\}$ is a finite or countable union of proper analytic subsets of X , i.e., of compact analytic curves or points. By hypothesis, S has no mass on the analytic curves in $\{x \in X, \nu(T, x) > 0\}$. Since S cannot have mass at any point of X , it has no mass on any countable set. We conclude that S has no mass on $\{x \in X, \nu(T, x) > 0\}$. Proposition 2.3 implies the result. \square

We see that the last result holds even when the wedge-product $T \wedge S$ is not defined. This somehow illustrates the flexibility of h-dimension and tangent currents in applications. Note also that when both T and S have positive mass on a curve Y , it is not difficult to see that the density between T and S is of the maximal h-dimension, i.e., 1.

3. EXPONENTIAL GROWTH RATE FOR ISOLATED PERIODIC POINTS

In this section, we will give the proofs of Theorems 1.1 and 1.2. We need to recall some properties of meromorphic maps on a compact Kähler manifold, see also [12, 13].

Consider a dominant meromorphic map $f : X \rightarrow X$ on a compact Kähler manifold of dimension k as in the beginning of Introduction. We will use the notations already

introduced there. The current $(f^n)^*(\omega^p)$ is positive closed and of bi-degree (p, p) . It is also an L^1 form which is smooth outside the indeterminacy set $I(f^n)$ of f^n . Recall that the mass of this current is comparable with the norm of $(f^n)^*$ on $H^{p,p}(X, \mathbb{C})$ and we can compute the dynamical degree of order p of f by

$$(3.1) \quad d_p(f) = \lim_{n \rightarrow \infty} \|(f^n)^*(\omega^p)\|^{1/n}.$$

We have the following lemma where we use the natural Kähler metric on $X \times X$ associated with the Kähler form $\pi_1^*(\omega) + \pi_2^*(\omega)$.

Lemma 3.1. *Let $\Gamma(f^n)$ denote the closure of the graph of f^n in $X \times X$ and $\|\Gamma(f^n)\|$ the mass of the current of integration on it. Then*

$$\lim_{n \rightarrow \infty} \|\Gamma(f^n)\|^{1/n} = e^{h_a(f)} = \max_{0 \leq p \leq k} d_p(f).$$

Proof. We have that

$$\|\Gamma(f^n)\| = \int_{\Gamma(f^n)} (\pi_1^*(\omega) + \pi_2^*(\omega))^k = \sum_{p=0}^k \binom{k}{p} \int_{\Gamma(f^n)} \pi_1^*(\omega^{k-p}) \wedge \pi_2^*(\omega^p).$$

Observe that π_1 defines a bijective map between a Zariski open set of $\Gamma(f^n)$ and a Zariski open set of X . Moreover, proper analytic subsets of $\Gamma(f^n)$ and of X have $2k$ -dimensional volume zero. Therefore, by pushing the integrals on $\Gamma(f^n)$ to X by π_1 , we see that the last sum of integrals is equal to

$$\sum_{p=0}^k \binom{k}{p} \int_X \omega^{k-p} \wedge (f^n)^*(\omega^p).$$

It is now easy to deduce the lemma from (3.1). \square

Note that by Wirtinger's theorem, the $2k$ -dimensional volume of $\Gamma(f^n)$ is equal to $\frac{1}{k!} \|\Gamma(f^n)\|$. So in the last lemma we can replace $\|\Gamma(f^n)\|$ by the volume of $\Gamma(f^n)$.

End of the proof of Theorem 1.1. Fix an $\epsilon > 0$. We only need to show that

$$\lim_{n \rightarrow \infty} e^{-n(h_a(f) + \epsilon)} P_n(f) = 0.$$

Define $d_n := e^{n(h_a(f) + \epsilon)}$. By Lemma 3.1, we have $d_n^{-1}[\Gamma(f^n)] \rightarrow 0$. We apply Proposition 2.2 for $\Gamma(f^n)$, $X \times X$, $\Delta, 0$ instead of Γ_n , X , V, T . The constant c there is then equal to 0 and the number δ_n is equal to $P_n(f)$. The desired property follows immediately. Note that Theorem 1.1 still remains true if f is a meromorphic correspondence, see [14] for related results. \square

Remark 3.2. When X is a projective manifold, we can also obtain the result by moving $\Gamma(f^n)$ in a family of (non-effective) cycles with controlled degrees. This approach can be generalized to maps defined over fields different from \mathbb{C} but cannot be used for Kähler manifolds, see also [29]. For Kähler manifolds, one can use the technique of regularization of currents in [12]. However, a mass control for intersections of currents is needed and this method does not yield better results.

We now prove Theorem 1.2. From now on, assume that X is a compact Kähler surface, i.e., $k = 2$. Let $f : X \rightarrow X$ be a meromorphic map as in this theorem which is algebraically stable and of minor topological degree. Write for simplicity $d_1 := d_1(f)$ and $d_2 := d_2(f)$. We have $d_1 > d_2 \geq 1$. We recall first some results by Diller-Favre [8] and Diller-Dujardin-Guedj [6].

There are two positive closed $(1, 1)$ -currents T_+ and T_- , called *dynamical Green currents*, such that $f^*(T^+) = d_1 T^+$, $f_*(T^-) = d_1(T^-)$ and $\{T^+\} \cup \{T^-\} = 1$. Moreover, the current T^+ has no mass on compact analytic curves in X . We will need the following proposition.

Proposition 3.3. *Let α and β be two smooth 2-forms on X , not necessarily positive nor closed. Define $c_\alpha := \langle T^-, \alpha \rangle$ and $c'_\beta := \langle T^+, \beta \rangle$. Then*

$$\lim_{n \rightarrow \infty} d_1^{-n} (f^n)^*(\alpha) = c_\alpha T^+ \quad \text{and} \quad \lim_{n \rightarrow \infty} d_1^{-n} (f^n)_*(\beta) = c'_\beta T^-.$$

Proof. Consider the first limit with α a $(2, 0)$ -form. Since T^- is of bi-degree $(1, 1)$, we have $c_\alpha = 0$. Let ϕ be any smooth test $(0, 2)$ -form on X . We have by the Cauchy-Schwarz inequality

$$\begin{aligned} |\langle d_1^{-n} (f^n)^*(\alpha), \phi \rangle| &\leq d_1^{-n} \left| \int (f^n)^*(\alpha) \wedge (f^n)^*(\bar{\alpha}) \right|^{1/2} \left| \int \phi \wedge \bar{\phi} \right|^{1/2} \\ &= d_1^{-n} \left| \int (f^n)^*(\alpha \wedge \bar{\alpha}) \right|^{1/2} \left| \int \phi \wedge \bar{\phi} \right|^{1/2}, \end{aligned}$$

where the integrals are taken on X outside the indeterminacy set of f^n . Since $\alpha \wedge \bar{\alpha}$ is a smooth form of maximal degree, the first integral in the last product is of order at most $O(d_2^n)$. Thus, the property $d_2 < d_1$ implies that $d_1^{-n} \langle (f^n)^*(\alpha), \phi \rangle$ tends to 0 as $n \rightarrow \infty$. It follows that $d_1^{-n} (f^n)^*(\alpha) \rightarrow 0$ and the lemma holds in this case because $c_\alpha = 0$.

Similarly, the lemma holds for α of bi-degree $(0, 2)$. We consider now the remaining case where α is of bi-degree $(1, 1)$. This case was already obtained in [6, Lemma 3.4] when α is the product of a smooth function with a Kähler form. It is easy to adapt their proof to any smooth $(1, 1)$ -form. We can also obtain the general case by writing α as a finite linear combination of forms which are products of smooth functions with Kähler forms. To see the last point, we can use a partition of unity to reduce the question to the local setting and then use a suitable family of Kähler forms to get a local frame for the vector bundle of $(1, 1)$ cotangent vectors of X .

We will now deduce the second limit in the proposition from the first one. Recall that the current $(f^n)_*(\beta)$ is defined as $(\pi_2)_*(\pi_1^*(\beta) \wedge [\Gamma(f^n)])$ and is defined by an L^1 form which is smooth outside an analytic set. For any smooth 2-form α , we have

$$\lim_{n \rightarrow \infty} \langle d_1^{-n} (f^n)_*(\beta), \alpha \rangle = \lim_{n \rightarrow \infty} \langle \beta, d_1^{-n} (f^n)^*(\alpha) \rangle = \langle c_\alpha T^+, \beta \rangle = c'_\beta c_\alpha = c'_\beta \langle T^-, \alpha \rangle.$$

This property holds for every smooth 2-form α . We deduce that $d_1^{-n} (f^n)_*(\beta) \rightarrow c'_\beta T^-$. \square

Proposition 3.4. *Let $\Gamma(f^n)$ denote the closure of the graph of f^n in $X \times X$. Then the sequence of positive closed $(2, 2)$ -currents $d_1^{-n} [\Gamma(f^n)]$ converges to $T^+ \otimes T^-$ as $n \rightarrow \infty$.*

Proof. Denote by (z^1, z^2) a general point in $X \times X$ with $z^1, z^2 \in X$. Consider smooth test $(2, 2)$ -forms Φ on $X \times X$ that can be written as

$$\Phi(z^1, z^2) = \beta(z^1) \wedge \alpha(z^2),$$

where β, α are smooth forms on X of bi-degrees respectively (p, q) and $(2-p, 2-q)$ with $0 \leq p, q \leq 2$. Since these forms span a dense vector space in the space of test $(2, 2)$ -forms, we only need to check that

$$(3.2) \quad d_1^{-n} \langle [\Gamma(f^n)], \Phi \rangle \rightarrow \langle T^+ \otimes T^-, \Phi \rangle.$$

Note that the left hand side of the last line is equal to

$$d_1^{-n} \langle [\Gamma(f^n)], \Phi \rangle = d_1^{-n} \int_{\Gamma(f^n)} \pi_1^*(\beta) \wedge \pi_2^*(\alpha) = d_1^{-n} \int_X \beta \wedge (f^n)^*(\alpha) = d_1^{-n} \int_X (f^n)_*(\beta) \wedge \alpha.$$

The right hand side of (3.2) is equal to $\langle T^+, \beta \rangle \langle T^-, \alpha \rangle$. So it vanishes except in the first case considered below.

Case 1. Assume that $p + q = 2$. We have

$$d_1^{-n} \langle [\Gamma(f^n)], \Phi \rangle = d_1^{-n} \int_X \beta \wedge (f^n)^*(\alpha) = \langle d_1^{-n} (f^n)^*(\alpha), \beta \rangle.$$

Proposition 3.3 implies that the last pairing converges to $\langle T^+, \beta \rangle \langle T^-, \alpha \rangle$. So (3.2) holds in this case.

Case 2. Assume that $p = q = 0$. So β is a function and α is a form of maximal degree. In particular, α defines a measure and therefore the mass of $(f^n)^*(\alpha)$ is of order at most $O(d_2^n)$. We then deduce that

$$d_1^{-n} \langle [\Gamma(f^n)], \Phi \rangle = \langle d_1^{-n} (f^n)^*(\alpha), \beta \rangle$$

is of order at most $O(d_1^{-n} d_2^n)$. The assumption that $d_2 < d_1$ implies that $\langle d_1^{-n} [\Gamma(f^n)], \Phi \rangle \rightarrow 0$ and we get (3.2) in this case.

Case 3. Assume that $p = q = 2$. So α is a function and β is a form of maximal degree. It follows that $(f^n)^*(\alpha)$ is a bounded function and we easily deduce that $\langle d_1^{-n} [\Gamma(f^n)], \Phi \rangle = O(d_1^{-n})$ and get (3.2) in this case.

Case 4. Assume that $p + q = 1$. We only consider the case $p = 1$ and $q = 0$ because the other case with $p = 0$ and $q = 1$ can be treated in the same way. So β is a $(1, 0)$ -form and α is a $(1, 2)$ -form. We can assume that $\alpha = \gamma \wedge \theta$, where γ is a $(0, 1)$ -form and θ is a Kähler form. Indeed, α can be written as a finite combination of such forms. By the Cauchy-Schwarz inequality, we have

$$d_1^{-n} |\langle [\Gamma(f^n)], \Phi \rangle| \leq d_1^{-n} |\langle [\Gamma(f^n)], \beta(z^1) \wedge \overline{\beta(z^1)} \wedge \theta(z^2) \rangle|^{1/2} |\langle [\Gamma(f^n)], \gamma(z^2) \wedge \overline{\gamma(z^2)} \wedge \theta(z^2) \rangle|^{1/2}.$$

We can apply the estimates in Cases 1 and 2 in order to bound the factors in the last product. We obtain that this product is of order $O(d_1^{-n} d_1^{n/2} d_2^{n/2})$. It follows that $d_1^{-n} \langle [\Gamma(f^n)], \Phi \rangle \rightarrow 0$ and we obtain (3.2) for this case.

Case 5. Assume in this remaining case that $p + q = 3$. We can also suppose that $p = 2$, $q = 1$ and β equal to the wedge-product of a $(1, 0)$ -form with a Kähler form. Then, this case can be treated exactly as in the last case using the estimates from Cases 1 and 3. \square

End of the proof of Theorem 1.2. By Proposition 3.4, we have $d_1^{-n} [\Gamma(f^n)] \rightarrow T^+ \otimes T^-$. Since T^+ has no mass on compact analytic curves in X , we can apply Corollary 2.4 to T^+, T^- instead of T and S . We obtain that the tangential h-dimension between $T^+ \otimes T^-$

along Δ is 0. Therefore, we can apply Proposition 2.2 to $X \times X, \Delta, d_1^n, \Gamma(f^n), T^+ \otimes T^-$ instead of X, V, d_n, Γ_n and T . The constant c in this proposition is equal to

$$\{T^+ \otimes T^-\} \smile \{\Delta\} = \{T^+\} \smile \{T^-\} = 1.$$

The constant δ_n there is equal to $P_n(f)$. Theorem 1.2 follows immediately. \square

Recall that the Lefschetz number $L(f^n)$ is the intersection number $\{\Gamma(f^n)\} \smile \{\Delta\}$. We have the following useful lemma.

Lemma 3.5. *Under the hypotheses of Theorem 1.2 and with the above notations, we have*

$$L(f^n) = d_1^n + o(d_1^n)$$

as $n \rightarrow \infty$.

Proof. Let $\alpha_{p,q,i}$ be smooth closed (p,q) -forms on X with $1 \leq i \leq \dim H^{p,q}(X, \mathbb{C})$ such that the classes $\{\alpha_{p,q,i}\}$ give a basis of the Hodge cohomology group $H^{p,q}(X, \mathbb{C})$. We also choose these forms so that the basis for $H^{p,q}(X, \mathbb{C})$ is dual to the basis for $H^{2-p,2-q}(X, \mathbb{C})$, i.e., $\{\alpha_{p,q,i}\} \smile \{\alpha_{2-p,2-q,j}\}$ is equal to 0 when $i \neq j$ and 1 if $i = j$. So the class of Δ in $H^{2,2}(X \times X, \mathbb{C})$ is equal to the class of

$$\sum_{p,q,i} \alpha_{p,q,i}(z^1) \wedge \alpha_{2-p,2-q,i}(z^2).$$

We then deduce that

$$L(f^n) = \sum_{p,q,i} \int_{\Gamma(f^n)} \alpha_{p,q,i}(z^1) \wedge \alpha_{2-p,2-q,i}(z^2) = \sum_{p,q,i} \int_X \alpha_{p,q,i} \wedge (f^n)^*(\alpha_{2-p,2-q,i}).$$

We have seen in the estimates from the proofs of Propositions 3.3 and 3.4 that the last integral is of order $o(d_1^n)$ when $(p,q) \neq (1,1)$.

The rest in the sum of integrals (those corresponding to $p = q = 1$) can be interpreted as the trace of $(f^n)^*$ on $H^{1,1}(X, \mathbb{C})$. Recall that since f is algebraically stable we have $(f^n)^* = (f^*)^n$ on $H^{1,1}(X, \mathbb{C})$. Moreover, it is known from [8] that d_1 is a simple eigenvalue of f^* acting on $H^{1,1}(X, \mathbb{C})$ and the other eigenvalues have moduli smaller than d_1 . It is now clear that the above trace of $(f^n)^*$ is equal to $d_1^n + o(d_1^n)$. The lemma follows. Note that a more precise estimate in terms of d_1, d_2 and the eigenvalues of f^* on $H^{1,1}(X, \mathbb{C})$ can be obtained. \square

4. BI-MEROMORPHIC MAPS ON COMPACT KÄHLER SURFACES

In this section, we will show that the theory of densities of currents may be combined with other arguments to get a lower estimate for the number of isolated periodic points. Namely, we will give here the proof of Theorem 1.4. From now on, we assume that $f : X \rightarrow X$ is an algebraically stable bi-meromorphic map on a compact Kähler surface as in that theorem.

We start with a short digression on Iwasaki-Uehara's version of Saito's fixed point formula in our setting. Denote by $I(f^n)$ and $I(f^{-n})$ the indeterminacy sets of f^n and f^{-n} respectively. These sets are finite. Recall that since f is algebraically stable, these two sets are disjoint, see e.g. [8, Proposition 1.13] and [9, Theorem 2.4]. So if x is a periodic point of period n of f , then either f^n is holomorphic near x and fixes the point x or the similar property holds for f^{-n} . Saito's local index function is the function $\nu_\bullet(f^n) : \text{Per}_n(f) \rightarrow \mathbb{N}$ defined in [21, Definition 3.5]. This function is 0 except at a finite

number of points. We only recall this notion in the case of isolated periodic points, since this is sufficient for the purpose here.

Let x be an isolated fixed point of f and for simplicity assume that f is holomorphic near x . Fix a local coordinate $z = (z_1, z_2)$ of X centered at x , i.e., $(z_1, z_2) = (0, 0)$ at x . In these coordinates, we can write

$$f(z) = (z_1 + h_1(z_1, z_2), z_2 + h_2(z_1, z_2)),$$

where h_1 and h_2 are holomorphic functions on an open neighborhood of 0 in \mathbb{C}^2 , vanishing at 0. Let \mathcal{O}_0 denote the ring of germs of holomorphic functions at $0 \in \mathbb{C}^2$. Let $\mathcal{I} \subset \mathcal{O}_0$ be the ideal generated by h_1 and h_2 . Since 0 is an isolated fixed point of f , the germ of analytic set defined by \mathcal{I} coincides with the single point $\{0\}$. In other words, \mathcal{I} is of complete intersection. By [21, Definition 3.5],

$$\nu_x(f) := \dim_{\mathbb{C}} \mathcal{O}_0 / \mathcal{I}.$$

The following lemma shows that Saito's index extends the notion of multiplicity for isolated periodic points.

Lemma 4.1. *If x is an isolated fixed point of f as above, then $\nu_x(f)$ is the multiplicity of the intersection $\Gamma(f) \cap \Delta$ at the point (x, x) .*

Proof. Let d denote the multiplicity of the intersection $\Gamma(f) \cap \Delta$ at the point (x, x) . Consider the germ of proper holomorphic map $h := (h_1, h_2) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$. This is a ramified covering of degree d from a neighborhood U of 0 in \mathbb{C}^2 to $h(U)$. The degree d represents the multiplicity of \mathcal{I} at 0, see e.g. [26]. Since \mathcal{I} is of complete intersection, this multiplicity is equal to the index $\nu_x(f)$ defined above, see e.g. [28]. The lemma follows. \square

Recall now the index of a fixed curve, i.e., a curve of fixed points. The notion can be easily extended to curves of periodic points. Let C be an irreducible analytic curve in X . We say that C is a *fixed curve* of f if $f(x) = x$ for all $x \in C \setminus I(f)$ or equivalently C is contained in $\text{Per}_1(f)$. Let $X_1(f)$ be the set of all fixed curves of f . Following [21, Definition 3.5], we can define the index function $\nu_{\bullet}(f) : X_1(f) \rightarrow \mathbb{N}$ as follows, see also [21, Lemma 6.1]. Let C be a curve in $X_1(f)$. Let x be a regular point of C which is not a point of indeterminacy for f . We can find local coordinates $z = (z_1, z_2)$ of X centered at x such that in these coordinates

$$f(z) = (z_1 + z_1^p h_1, z_2 + z_1^q h_2),$$

where $p, q \in \mathbb{N}_+$, h_1, h_2 are holomorphic on a neighborhood of 0, and $h_1(0, z_2), h_2(0, z_2)$ do not vanish identically. Define

$$\nu_C(f) := \min(p, q).$$

This notion is independent of the choices of coordinates. Observe that except for a finite number of points x , we have $h_1(0) \neq 0$ and $h_2(0) \neq 0$, and therefore, reducing the neighborhood of x allows us to assume that h_1 and h_2 are nowhere vanishing.

The following lemma relates $\nu_C(f)$ to the density of the current $[\Gamma(f)]$ along Δ , see Section 2 for notations. Recall that we identify Δ with X in a canonical way.

Lemma 4.2. *The tangential h -dimension of $[\Gamma(f)]$ along Δ is at most 1 and we have*

$$\kappa_1^\Delta([\Gamma(f)]) = \sum_{C \in X_1(f)} \nu_C(f) \{C\}.$$

Proof. For simplicity, write $\Gamma := \Gamma(f)$, $X_1 := X_1(f)$, $\nu_C := \nu_C(f)$ and $\kappa_j := \kappa_j^\Delta([\Gamma(f)])$. We will apply the theory of densities of currents to $X \times X$, $[\Gamma(f)]$, Δ instead of X, T, V , see Section 2 for details. We will use notations similar to those in that section. Denote by \overline{E} the natural compactification of the normal vector bundle E of Δ in $X \times X$ and $\pi : \overline{E} \rightarrow \Delta$ the canonical projection. Consider an arbitrary tangent current R of $[\Gamma]$ along Δ . The map $A_\lambda : \overline{E} \rightarrow \overline{E}$ is the multiplication by λ on the fibers of \overline{E} . The first assertion in the lemma is automatically true and we only have to prove the second one.

Recall that if $-h$ is the tautological $(1, 1)$ -class of \overline{E} , we have the following unique decomposition $\{R\} = \pi^*(\kappa_1) \smile h + \pi^*(\kappa_0)$, see (2.1). We have to show that

$$\kappa_1 = \sum_{C \in X_1} \nu_C \{C\}.$$

Let α be an arbitrary closed $(1, 1)$ -form on Δ . By Poincaré's duality, it is enough to check that

$$(4.1) \quad \kappa_1 \smile \{\alpha\} = \sum_{C \in X_1} \nu_C \{C\} \smile \{\alpha\}.$$

We can assume that α is a Kähler form because the classes of such forms spans $H^{1,1}(X, \mathbb{C})$. So the right hand side of the last identity is the mass of the positive measure $\sum \nu_C [C] \wedge \alpha$. Denote this mass by m .

Observe that $\{R \wedge \pi^*(\alpha)\} = \pi^*(\kappa_1 \smile \{\alpha\}) \smile h$. As we have seen in the discussion before (2.1), this decomposition is unique. Thus, identity (4.1) is equivalent to

$$(4.2) \quad \{R \wedge \pi^*(\alpha)\} = mH,$$

where H denotes the restriction of h to a fiber of \overline{E} . Note that H is the class of a hyperplane of a fiber of \overline{E} .

Consider a point $(x, x) \in \Delta$ with $x \in X$. Let $z = (z_1, z_2)$ denote a local coordinate system in X centered at x , which identifies a neighborhood of x with a neighborhood V_0 of 0 in \mathbb{C}^2 . It also induces naturally a local coordinate system (z, z') with $z' = (z'_1, z'_2)$ of $X \times X$ centered at (x, x) . The diagonal Δ is given by $z = z'$ in these coordinates. Define $w = (w_1, w_2) = z' - z$. So (z, w) is a new local coordinate system centered at (x, x) and Δ is given by $w = 0$. We can identify a neighborhood of (x, x) with some neighborhood U of $V_0 \times \{0\}$ in $V_0 \times \mathbb{C}^2$. Here, Δ is identified with $V_0 \times \{0\}$, the normal bundle E with $V_0 \times \mathbb{C}^2$, and the projection π with the natural projection from $V_0 \times \mathbb{C}^2$ to V_0 . The map A_λ is then given by $A_\lambda(z, w) = (z, \lambda w)$. We also choose the admissible map τ to be the identity map as the standard local setting described in Section 2.

We see in this picture that $(A_\lambda)_*[\Gamma]$ is given by the analytic subset $A_\lambda(\Gamma)$ of pure dimension 2 in $A_\lambda(U)$. The last set is an open set converging to $V_0 \times \mathbb{C}^2$ as $\lambda \rightarrow \infty$. Given any compact set in $V_0 \times \mathbb{C}^2$, the theory of densities insures that the volume of $A_\lambda(\Gamma)$ (or equivalently the mass of the associated current) in K is bounded when $\lambda \rightarrow \infty$. This allows us to extract a converging subsequence from the family of currents $(A_\lambda)_*[\Gamma]$ and get a tangent current R , see Section 2. We deduce that R is given by a finite linear combination $\sum m_i \Gamma_i$ of irreducible sub-varieties Γ_i of dimension 2 in E , where m_i are

positive integers. We refer to King [25] for details on currents defined by positive chains of varieties. The varieties Γ_i are invariant by A_λ for every λ because R satisfies this property.

If x is not a fixed point, we can choose U small enough so that it has no intersection with Γ . In this case, we see that $R = 0$ in $V_0 \times \mathbb{C}^2$. In contrast, if x is a fixed point, then $A_\lambda(\Gamma)$ contains (x, x) and therefore some Γ_i contains this point. We conclude that the union of $\Gamma_i \cap \Delta$ is exactly the set of points (x, x) with $x \in \text{Per}_1(f)$. Consider first an index i such that $\Gamma_i \cap \Delta$ is finite. Since Γ_i is irreducible, of dimension 2, and invariant by A_λ , we deduce that Γ_i is a fiber of E . Therefore, we get $[\Gamma_i] \wedge \pi^*(\alpha) = 0$. So such a Γ_i does not contribute to the left hand side of identity (4.2). We then deduce that $R \wedge \pi^*(\alpha)$ has no mass on each fiber of π . It remains now to analyze the situation near a point (x, x) of Δ where x is a generic point on a curve C in X_1 .

Recall that tangent currents do not depend on the choice of coordinates. So we can choose z as in the discussion just before Lemma 4.2. With this choice, the variety $A_\lambda(\Gamma)$ is given by

$$w_1 = \lambda z_1^p h_1(z_1, z_2) \quad \text{and} \quad w_2 = \lambda z_1^q h_2(z_1, z_2).$$

Consider first the case where $p = q$. The last variety is a ramified covering of degree p over a neighborhood of 0 in the plane (z_2, w_1) . Taking $\lambda \rightarrow \infty$, we see that the current $(A_\lambda)_*([\Gamma])$ converges to p times the current of integration on the variety

$$z_1 = 0 \quad \text{and} \quad w_2 = \frac{h_2(0, z_2)}{h_1(0, z_2)} w_1.$$

When $p < q$, the limit is p times the current of integration on $z_1 = w_2 = 0$ and when $p > q$ the limit is q times the current of integration on $z_1 = w_1 = 0$. In any case, we see that $R \wedge \pi^*(\alpha)$ restricted to $\pi^{-1}(V_0)$ can be decomposed into currents of integration on hyperplanes of the fibers of π . Moreover, its cohomology class is equal to the mass of $\nu_C \cdot [C] \wedge \alpha$ in V_0 times H . The desired identity (4.2) follows. \square

Recall that the set $X_1(f)$ of all fixed curves can be divided into two disjoint families, namely, into what Saito called *the curves of type I* and *type II*:

$$X_1(f) = X_{\text{I}}(f) \sqcup X_{\text{II}}(f).$$

With the notation as above, the curve C is of type I when $p \leq q$ and of type II when $p > q$. One can check that the definition does not depend on the choice of coordinates. Roughly speaking, C is of type II if near a generic point of C , the map f moves slightly a point to another along a tangential direction to C . We refer the reader to [21, Def. 3.7 and Lemma 6.1] for more details.

Now we are in the position to state Iwasaki-Uehara's version of Saito's fixed point formula. This is a key argument in the proof of Theorem 1.4. Note that these authors state their result for projective surfaces but their proof also works for Kähler surfaces. Of course, this result also applies to f^n with $n \in \mathbb{Z}$.

Theorem 4.3. (Saito [27], Iwasaki-Uehara [21]) *Under the above hypotheses and notations, the Lefschetz number $L(f)$ can be expressed as*

$$L(f) = \sum_{x \in \text{Per}_1(f)} \nu_x(f) + \sum_{C \in X_{\text{I}}(f)} \chi_C \cdot \nu_C(f) + \sum_{C \in X_{\text{II}}(f)} \tau_C \cdot \nu_C(f),$$

where χ_C is the Euler characteristic of the normalization of C and τ_C is the self-intersection number of C .

End of the proof of Theorem 1.4. By Theorem 4.3, applied to f^n , we have

$$(4.3) \quad L(f^n) = \sum_{x \in \text{Per}_n(f)} \nu_x(f^n) + \sum_{C \in X_I(f^n)} \chi_C \cdot \nu_C(f^n) + \sum_{C \in X_{II}(f^n)} \tau_C \cdot \nu_C(f^n).$$

We need to show that the first term in the right hand side is equal to $d_1(f)^n + o(d_1(f)^n)$. By Lemma 3.5, we only need to check that the last two terms in the previous identity is equal to $o(d_1(f)^n)$.

Consider the first term with $C \in X_I(f^n)$. A result of Diller-Jackson-Sommese [9, Theorem 3.6] says that the genus of such a curve C is zero or one, that is, χ_C is 2 or 0. Therefore, we have

$$(4.4) \quad 0 \leq \sum_{C \in X_I(f^n)} \chi_C \cdot \nu_C(f^n) \leq 2 \sum_{C \in X_I(f^n)} \nu_C(f^n).$$

For simplicity, denote by Γ_n the closure of the graph of f^n in $X \times X$. By Lemma 4.2, the tangential h-dimension of $[\Gamma_n]$ along Δ is at most equal to 1 and

$$\kappa_1^\Delta([\Gamma_n]) = \sum_{C \in X_I(f^n)} \nu_C(f^n) \{C\}.$$

Moreover, we know by Proposition 3.4 that $d_1(f)^{-n}[\Gamma_n] \rightarrow T^+ \otimes T^-$ and by Corollary 2.4 that $T^+ \otimes T^-$ has the tangential h-dimension 0 along Δ . Consequently, applying Proposition 2.1 yields

$$d_1(f)^{-n} \kappa_1^\Delta([\Gamma_n]) \rightarrow 0 \quad \text{or equivalently} \quad \sum_{C \in X_I(f^n)} \nu_C(f^n) \{C\} = o(d_1(f)^n).$$

Thus,

$$(4.5) \quad \sum_{C \in X_I(f^n)} \nu_C(f^n) \{C\} \sim \{\omega\} = o(d_1(f)^n),$$

where ω is any fixed Kähler form on X . Note that $\{C\} \sim \{\omega\}$ is 2 times the area of C with respect to the Kähler metric ω . Therefore, it is bounded below by a positive constant independent of C . This together with (4.4) implies that the second term in the right hand side of (4.3) is equal to $o(d_1(f)^n)$.

It remains to prove a similar bound for the last term in (4.3). For this purpose, we have two different approaches.

First approach. We know by Iwasaki-Uehara [21, Theorem 2.4] that $\mathcal{X} := \cup_{n=1}^\infty X_{II}(f^n)$ is a finite set. For every $C \in \mathcal{X}$, there is a smallest integer k_C such that $C \in X_{II}(f^{k_C})$. Moreover, if $C \in X_{II}(f^n)$, [21, Theorem 2.1] tells us that $\nu_C(f^{kn}) = \nu_C(f^n)$ for all $k \geq 1$. Consequently, for every $n \geq 1$ and $C \in X_{II}(f^n)$, we have

$$\nu_C(f^n) = \nu_C(f^{nk_C}) = \nu_C(f^{k_C}).$$

Thus,

$$\left| \sum_{C \in X_{II}(f^n)} \tau_C \cdot \nu_C(f^n) \right| \leq \sum_{C \in \mathcal{X}} |\tau_C| \cdot \nu_C(f^{k_C}).$$

The last sum is a constant independent of n . The desired estimate follows.

Second approach. By Diller-Jackson-Sommese [9, Theorem 3.6], the arithmetic genus $g(C)$ of $C \in X_1(f^n)$ is either zero or one. By the genus formula (e.g. Section 11, Chapter 2 in [2]), $g(C)$ and the self-intersection τ_C of a curve C in the surface X are related via

$$\tau_C = 2g(C) - 2 - K_X \cdot \{C\},$$

where K_X is the canonical class of X . Consequently, there is a constant $M > 0$ independent of C such that

$$|\tau_C| \leq M \cdot \{C\} \cdot \{\omega\}.$$

Therefore,

$$\left| \sum_{C \in X_{II}(f^n)} \tau_C \cdot \nu_C(f^n) \right| \leq M \sum_{C \in X_1(f^n)} \nu_C(f^n) \cdot \{C\} \cdot \{\omega\}.$$

We then conclude the proof using estimate (4.5). \square

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